

AUTOMORPHISMS OF \mathbb{C}^3 COMMUTING WITH A \mathbb{C}^+ -ACTION

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ABSTRACT. Let ρ be an algebraic action of the additive group \mathbb{C}^+ on the three-dimensional affine space \mathbb{C}^3 . We describe the group $\text{Cent}(\rho)$ of polynomial automorphisms of \mathbb{C}^3 that commute with ρ . A particular emphasis lies in the description of the automorphisms in $\text{Cent}(\rho)$ coming from algebraic \mathbb{C}^+ -actions. As an application we prove that the automorphisms in $\text{Cent}(\rho)$ that are the identity on the algebraic quotient of ρ form a characteristic subgroup of $\text{Cent}(\rho)$.

1. INTRODUCTION

Let X be an affine algebraic variety. A classification of the algebraic \mathbb{C}^+ -actions on X up to conjugacy in the automorphism group $\text{Aut}(X)$ is only known for a few varieties X . For example when $X = \mathbb{C}^2$ we have a classification: every \mathbb{C}^+ -action is a modified translation up to conjugacy, i.e. an action of the form $t \cdot (x, y) = (x + td(y), y)$ for a suitable polynomial d (see [Ren68]). In contrast to the two-dimensional case, there is no classification known for the \mathbb{C}^+ -actions on \mathbb{C}^3 . As a first step towards a classification, we study in this article the centralizer of a \mathbb{C}^+ -action in $\text{Aut}(\mathbb{C}^3)$, i.e. the group of automorphisms that commute with the \mathbb{C}^+ -action.

It is known that there is a bijective correspondence of algebraic \mathbb{C}^+ -actions on \mathbb{C}^n and locally nilpotent derivations of the polynomial ring in n -variables $\mathcal{O}(\mathbb{C}^n) = \mathbb{C}[x_1, \dots, x_n]$, i.e. \mathbb{C} -derivations of $\mathcal{O}(\mathbb{C}^n)$ such that for every $f \in \mathcal{O}(\mathbb{C}^n)$ there exists an integer $n = n(f)$ such that $D^n(f) = 0$ (see [Fre06, sec. 1.5]). The correspondence is given by the exponential, i.e. the \mathbb{C}^+ -action corresponding to a locally nilpotent derivation D is given by

$$t \cdot (x_1, \dots, x_n) = \text{Exp}(tD)(x_1, \dots, x_n) = \left(\sum_{i=0}^{\infty} \frac{t^i D^i(x_1)}{i!}, \dots, \sum_{i=0}^{\infty} \frac{t^i D^i(x_n)}{i!} \right).$$

Definition 1.0.1. An automorphism of the form $\text{Exp}(D) \in \text{Aut}(\mathbb{C}^n)$ is called *unipotent* where D is a locally nilpotent derivation of $\mathcal{O}(\mathbb{C}^n)$. For a subset $S \subseteq \text{Aut}(\mathbb{C}^n)$ we denote by S_u the unipotent automorphisms in S .

Remark 1.0.2. An algebraic \mathbb{C}^+ -action ρ on \mathbb{C}^n is uniquely determined by the unipotent automorphism $\rho_1 = ((x_1, \dots, x_n) \mapsto \rho(1, x_1, \dots, x_n))$ of \mathbb{C}^n and every unipotent automorphism of \mathbb{C}^n can be constructed in this way. Thus $\rho \mapsto \rho_1$ is a bijection between algebraic \mathbb{C}^+ -actions on \mathbb{C}^n and unipotent automorphisms of \mathbb{C}^n . Moreover, the centralizer of ρ is the same as the centralizer of ρ_1 .

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Definition 1.0.3. Let $\mathbf{u} = \text{Exp}(D)$ be a unipotent automorphism of \mathbb{C}^n and let $f \in \ker D$. Then fD is a locally nilpotent derivation and we call $\text{Exp}(fD)$ a *modification* of \mathbf{u} . We then denote

$$f \cdot \mathbf{u} = \text{Exp}(fD).$$

We call \mathbf{u} irreducible, if $\mathbf{u} \neq \text{id}$ and the following holds: if $\mathbf{u} = f \cdot \mathbf{u}'$ for some unipotent $\mathbf{u}' \in \text{Aut}(\mathbb{C}^n)$ and some $f \in \ker D$, then $f \in \mathbb{C}^*$. If $\mathbf{u} \neq \text{id}$, then there exists an irreducible $\mathbf{u}' \in \text{Aut}(\mathbb{C}^n)$ such that $\mathbf{u} = d \cdot \mathbf{u}'$ for some $d \in \ker D$ and \mathbf{u}' is unique up to a modification by some element in \mathbb{C}^* . We call then $\mathbf{u} = d \cdot \mathbf{u}'$ a *standard decomposition*.

Remark 1.0.4. If $\mathbf{u} = \text{Exp}(D) \in \text{Aut}(\mathbb{C}^n)$ is unipotent, then the ring of \mathbf{u} -invariant polynomials $\mathcal{O}(\mathbb{C}^n)^{\mathbf{u}}$ is equal to $\ker D$. The group of modifications of \mathbf{u} we denote then by $\mathcal{O}(\mathbb{C}^n)^{\mathbf{u}} \cdot \mathbf{u}$.

In dimension $n = 2$, SHMUEL FRIEDLAND and JOHN MILNOR proved that every automorphism of \mathbb{C}^2 is conjugate to a composition of generalized HÉNON maps or to a triangular automorphism (cf. [FM89, Theorem 2.6]). In the first case, STÉPHANE LAMY showed that the centralizer of such an automorphism is isomorphic to a semi-direct product of \mathbb{Z} with a finite cyclic group \mathbb{Z}_q (cf. [Lam01, Proposition 4.8]). In the second case, assuming in addition that the automorphism is unipotent, it has the form $\mathbf{u}(x, y) = (x + d(y), y)$. Thus, $\mathbf{u} = d \cdot \mathbf{u}'$ is a standard decomposition where $\mathbf{u}'(x, y) = (x + 1, y)$. One can check that the centralizer $\text{Cent}(\mathbf{u})$ fits in the following split short exact sequence

$$1 \rightarrow \mathcal{O}(\mathbb{C}^2)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \text{Cent}(\mathbf{u}) \twoheadrightarrow \text{Aut}(\mathbb{C}, \Gamma) \rightarrow 1 \quad (1)$$

where $\text{Aut}(\mathbb{C}, \Gamma)$ denotes the automorphisms of \mathbb{C} preserving the principal divisor $\Gamma = \text{div}(d)$ in \mathbb{C} .

In dimension $n = 3$, CINZIA BISI proved that any automorphism \mathbf{g} that commutes with a so-called regular automorphism \mathbf{f} satisfies $\mathbf{g}^m = \mathbf{f}^k$ for certain integers k, m (cf. [Bis08, Main Theorem 1.1]). As a counterpart to the regular automorphisms, one can regard the unipotent automorphisms (a regular automorphism is always algebraically stable and thus can not be unipotent). The work of DAVID FINSTON and SEBASTIAN WALCHER [FW97] can be seen as a first step in the study of the centralizer of a unipotent automorphism. They explore the centralizer of a triangulable (locally nilpotent) derivation inside the algebra of all derivations of the polynomial ring $\mathcal{O}(\mathbb{C}^3)$.

2. STATEMENT OF THE MAIN RESULTS

Let us recall briefly some notion and facts of the theory of ind-groups (see [Kum02, ch. IV] for an introduction). A group G is called an *ind-group* if it is endowed with a filtration by affine varieties $G_1 \subseteq G_2 \subseteq \dots$, each one closed in the next, such that $G = \bigcup_{i=1}^{\infty} G_i$ and such that the map $G \times G \rightarrow G$, $(x, y) \mapsto x \cdot y^{-1}$ is a morphism of ind-varieties. We then write $G = \varinjlim G_i$. For example, $\text{Aut}(\mathbb{C}^n)$ is an ind-group with the filtration $\text{Aut}(\mathbb{C}^n)_1 \subseteq \text{Aut}(\mathbb{C}^n)_2 \subseteq \dots$ where $\text{Aut}(\mathbb{C}^n)_i$ is the set of all automorphisms of degree $\leq i$ (see [BCW82]). We endow an ind-group $G = \varinjlim G_i$ with the following topology: a subset $X \subseteq G$ is closed if and only if $X \cap G_i$ is closed in G_i for each i . A subgroup H of an ind-group $G = \varinjlim G_i$ is called *algebraic*, if it is a closed subset of some G_i . We call an element $g \in G$ *algebraic*, if the closure of the cyclic group $\langle g \rangle$ is an algebraic subgroup of G .

In order to state our main results we introduce some notation and recall some facts about \mathbb{C}^+ -actions on \mathbb{C}^3 . Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent. We denote by $\pi: \mathbb{C}^3 \rightarrow \mathbb{C}^3 // \mathbb{C}^+$ the algebraic quotient of the \mathbb{C}^+ -action $(t, v) \mapsto (t \cdot \mathbf{u})(v)$, i.e. π corresponds to the inclusion $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}} \subseteq \mathcal{O}(\mathbb{C}^3)$.

Definition 2.0.5. Let $\mathbf{u} = \text{Exp}(D) \in \text{Aut}(\mathbb{C}^3)$ be unipotent. We call the ideal $\text{im } D \cap \ker D$ of $\ker D$ the *plinth ideal*. By [DK09, Theorem 1] the plinth ideal is principal. We fix some generator of it and denote it by a . We denote further by

$$\Gamma = \text{div}(a)$$

the principal divisor in $\mathbb{C}^3 // \mathbb{C}^+$ corresponding to a and call it the *plinth divisor* of D (respectively of \mathbf{u}). As Γ is completely determined by the closed subscheme $V(a) \subseteq \mathbb{C}^3 // \mathbb{C}^+$, we can and will identify this scheme with Γ .

Remark 2.0.6. By MIYANISHI's Theorem (cf. [Fre06, Theorem 5.1]), the algebraic quotient $\mathbb{C}^3 // \mathbb{C}^+$ is isomorphic to \mathbb{C}^2 . The restriction $\pi|_{\mathbb{C}^3 \setminus \pi^{-1}(\Gamma)}: \mathbb{C}^3 \setminus \pi^{-1}(\Gamma) \rightarrow \mathbb{C}^2 \setminus \Gamma$ is a trivial principal \mathbb{C}^+ -bundle (cf. [Fre06, Principle 11]).

Definition 2.0.7. Let $\Gamma \subseteq \mathbb{C}^2$ be an effective divisor. We denote by $\text{Aut}(\mathbb{C}^2, \Gamma)$ the subgroup of all $\mathbf{g} \in \text{Aut}(\mathbb{C}^2)$ such that the scheme-theoretic image $\mathbf{g}(\Gamma)$ is again Γ and we denote by $\text{Iner}(\mathbb{C}^2, \Gamma)$ the subgroup of all $\mathbf{g} \in \text{Aut}(\mathbb{C}^2, \Gamma)$ such that the pullback to Γ is the identity. If $\Gamma' \subseteq \mathbb{C}^2$ is another effective divisor, then we denote the intersection $\text{Aut}(\mathbb{C}^2, \Gamma) \cap \text{Aut}(\mathbb{C}^2, \Gamma')$ by $\text{Aut}(\mathbb{C}^2, \Gamma, \Gamma')$.

Definition 2.0.8. Let $\Gamma = \sum_i n_i \Gamma_i$ be an effective divisor in \mathbb{C}^2 . We call Γ a *fence*, if $\Gamma_i \simeq \mathbb{C}$ for all i and the Γ_i are pairwise disjoint.

Let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition of a unipotent $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ and let Γ, Γ' be the plinth divisors of \mathbf{u}, \mathbf{u}' respectively. We have an induced action of the centralizer $\text{Cent}(\mathbf{u})$ on the algebraic quotient $\mathbb{C}^3 // \mathbb{C}^+$ that preserves Γ and also Γ' . This implies that there is a sequence of ind-groups

$$1 \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \text{Cent}(\mathbf{u}) \xrightarrow{p} \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma, \Gamma'), \quad (2)$$

which is exact by Proposition 5.1.1. In contrast to the two-dimensional case (see (1)), the homomorphism p is in general not surjective (see [Sta13, Proposition 1]). Thus, in order to study $\text{Cent}(\mathbf{u})$ we have to understand the image of p .

The description of $\text{Cent}(\mathbf{u})$ is special in the case when \mathbf{u} is a *translation*, i.e. $\mathbf{u}(x, y, z) = (x+1, y, z)$ for suitable coordinates (x, y, z) . Note that \mathbf{u} is a translation if and only if its plinth divisor Γ is empty.

2.1. Structure theorems for $\text{Cent}(\mathbf{u})$.

Proposition A (cf. Proposition 5.2.1). *Let $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be a translation. Then*

$$1 \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}} \cdot \mathbf{u} \hookrightarrow \text{Cent}(\mathbf{u}) \rightarrow \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+) \rightarrow 1$$

is a split short exact sequence of ind-groups.

Theorem B (cf. Theorem 5.6.1, 5.7.1 and Corollary 5.7.2). *Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and not a translation. Then*

- i) *All elements in $\text{Cent}(\mathbf{u})$ are algebraic.*
- ii) *The set of unipotent elements $\text{Cent}(\mathbf{u})_u \subseteq \text{Cent}(\mathbf{u})$ is a closed normal subgroup.*
- iii) *There exists an algebraic subgroup $R \subseteq \text{Cent}(\mathbf{u})$ consisting only of semi-simple elements such that $\text{Cent}(\mathbf{u}) \simeq \text{Cent}(\mathbf{u})_u \rtimes R$ as ind-groups.*

Theorem C (cf. Proposition 5.5.1 and Theorem 5.7.1). *Let $\text{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent, not a translation and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. Let Γ and Γ' be the plinth divisors of \mathbf{u} and \mathbf{u}' respectively. Then, the sequence induced by (2)*

$$1 \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \text{Cent}(\mathbf{u})_u \xrightarrow{p} \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma) \cap \text{Iner}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma')_u \rightarrow 1$$

is a split short exact sequence of ind-groups. Moreover, there exists an irreducible unipotent $\mathbf{e} \in \text{Aut}(\mathbb{C}^3)$ such that the restriction of p to $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e}$ is an isomorphism of ind-groups ($\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle}$ is the subring of \mathbf{e} -invariant polynomials inside $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}}$).

Let us give some explanation of the last result. If Γ is not a fence, then the underlying variety cannot be a union of orbits of a non-trivial \mathbb{C}^+ -action on \mathbb{C}^2 . Hence all unipotent automorphisms of $\text{Cent}(\mathbf{u})$ induce the identity on the algebraic quotient and $\text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma) \cap \text{Iner}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma')_u$ is trivial. Thus the result follows from (2). So let us assume that Γ is a non-empty fence. There exists a proper non-empty open subset $U \subseteq \mathbb{C}$ such that we have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{C}^3 & \supseteq & \mathbb{C}^3 \setminus \pi^{-1}(\Gamma) & \simeq & (U \times \mathbb{C}) \times \mathbb{C} \\ \pi \downarrow & & \downarrow & & \downarrow \text{pr} \\ \mathbb{C}^2 & \supseteq & \mathbb{C}^2 \setminus \Gamma & \simeq & U \times \mathbb{C} \end{array}$$

where pr denotes the projection onto the first two factors (see Proposition 3.0.3). There exist coordinates (u, v, w) of $(U \times \mathbb{C}) \times \mathbb{C}$ such that the automorphism induced by \mathbf{u} on $(U \times \mathbb{C}) \times \mathbb{C}$ is given by $(u, v, w) \mapsto (u, v, w + 1)$. The unipotent automorphism $(u, v, w) \mapsto (u, v + 1, w)$ of $(U \times \mathbb{C}) \times \mathbb{C}$ extends to an irreducible unipotent automorphism \mathbf{e} on \mathbb{C}^3 that commutes with \mathbf{u} (see subsec. 5.3). Thus, $\text{Cent}(\mathbf{u})_u$ contains $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e}$ beside $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$. The difficulty lies in proving that $\text{Cent}(\mathbf{u})_u$ is a group, generated by $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e}$ and $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$.

2.2. Applications. We present two applications of the structure theorems. The first one concerns abstract automorphisms of $\text{Cent}(\mathbf{u})$.

Proposition D (cf. Proposition 6.0.4). *Let $\text{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent, not a translation and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. Then, the subgroup $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \subseteq \text{Cent}(\mathbf{u})$ is characteristic, i.e. it is invariant under all abstract group automorphisms of $\text{Cent}(\mathbf{u})$.*

The second application concerns the plinth divisor Γ . The reduced scheme Γ_{red} has the following geometric description: The complement of Γ_{red} is the maximal open subset of $\mathbb{C}^3 // \mathbb{C}^+$ such that the algebraic quotient $\pi: \mathbb{C}^3 \rightarrow \mathbb{C}^3 // \mathbb{C}^+$ is a locally trivial principal \mathbb{C}^+ -bundle over it (see [DK14, Proposition 5.4]). So far - to the author's knowledge - there is no geometric description of the *scheme* Γ . But in the case when Γ is a non-empty fence and \mathbf{u} is irreducible we can give one.

Proposition E (cf. Proposition 6.0.3). *Let $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and irreducible, and assume that Γ is a non-empty fence. Then $\Gamma \subseteq \mathbb{C}^3 // \mathbb{C}^+$ is the largest closed subscheme fixed by $\text{Cent}(\mathbf{u})_u$.*

3. AUTOMORPHISMS OF \mathbb{C}^2 THAT PRESERVE A DIVISOR

Let $\Gamma \subseteq \mathbb{C}^2$ be an effective divisor. We have an exact sequence

$$1 \rightarrow \text{Iner}(\mathbb{C}^2, \Gamma) \hookrightarrow \text{Aut}(\mathbb{C}^2, \Gamma) \rightarrow \text{Aut}(\Gamma)$$

(see Definition 2.0.7). The next result uses heavily the main result in [BS13].

Proposition 3.0.1. *Let Γ be a non-trivial effective divisor of \mathbb{C}^2 . Then*

- i) *The subgroup $\text{Aut}(\mathbb{C}^2, \Gamma) \subseteq \text{Aut}(\mathbb{C}^2)$ is closed and all elements of $\text{Aut}(\mathbb{C}^2, \Gamma)$ are algebraic.*
- ii) *The following statements are equivalent*
 - a) *Γ is a fence*
 - b) *$\text{Aut}(\mathbb{C}^2, \Gamma)$ contains unipotent automorphisms $\neq \text{id}$.*
 - c) *$\text{Iner}(\mathbb{C}^2, \Gamma) \neq \{\text{id}\}$*
 - d) *$\text{Aut}(\mathbb{C}^2, \Gamma)$ is not an algebraic group*

For the proof of this proposition we recall some facts about $\text{Aut}(\mathbb{C}^2)$. The next result is a direct consequence of [BS13, Theorem 1].

Theorem 3.0.2. *An automorphism of \mathbb{C}^2 that preserves an algebraic curve is conjugate to a triangular automorphism.*

In the next result we prove a slightly more general version of the ABHYANKAR-MOH-SUZUKI-Theorem which says that all closed embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^2$ are equivalent up to automorphisms of \mathbb{C}^2 (see [AM75]).

Proposition 3.0.3. *Let Γ be a fence in \mathbb{C}^2 and let $F \subseteq \mathbb{C}$ be a closed 0-dimensional subscheme such that $\Gamma \simeq F \times \mathbb{C}$. Then there exists an automorphism of \mathbb{C}^2 that maps Γ onto $F \times \mathbb{C}$ (scheme-theoretically).*

Proof. Clearly, we can assume that $\Gamma \neq \emptyset$. Moreover, we can easily reduce to the case, where Γ is a reduced scheme. Let Γ_i , $i \in I$ be the irreducible components of Γ . Let $i_0 \in I$ be fixed. By the ABHYANKAR-MOH-SUZUKI-Theorem, there exists a trivial \mathbb{C} -bundle $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ such that Γ_{i_0} is a fiber of f . Now, if the restriction $f|_{\Gamma_i}: \Gamma_i \rightarrow \mathbb{C}$ is non-constant, then it is surjective, since $\Gamma_i \simeq \mathbb{C}$. But this implies that $\Gamma_i \cap \Gamma_{i_0} \neq \emptyset$, a contradiction. Thus every Γ_i is a fiber of f . This implies the proposition. \square

Proof of Proposition 3.0.1.

- i) Assume that $\Gamma = \text{div}(a)$ for some non-zero $a \in \mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x, y]$. For $(f_1, f_2) \in \mathbb{C}[x, y]^2$ we denote by $a_{ij}(f_1, f_2)$ the coefficient of the monomial $x^i y^j$ in the polynomial $a(f_1, f_2)$. The subgroup $\text{Aut}(\mathbb{C}^2, \Gamma)$ of $\text{Aut}(\mathbb{C}^2)$ is defined by the equations

$$a_{ij}(f_1, f_2)a_{kl}(x, y) = a_{kl}(f_1, f_2)a_{ij}(x, y) \quad \text{for all pairs } (i, j), (k, l).$$

This proves the first statement.

Let $\mathbf{g} \in \text{Aut}(\mathbb{C}^2, \Gamma)$. By Theorem 3.0.2, \mathbf{g} is conjugate to a triangular automorphism and hence \mathbf{g} is algebraic. This proves the second statement.

- ii) a) \Rightarrow b): This follows immediately from Proposition 3.0.3.

b) \Rightarrow c): Let $\mathbf{g} \in \text{Aut}(\mathbb{C}^2, \Gamma)$ be a unipotent automorphism $\neq \text{id}$. Choose some $a \in \mathcal{O}(\mathbb{C}^2)$ such that $\Gamma = \text{div}(a)$. As \mathbf{g} preserves Γ , it follows that a is a semi-invariant for the \mathbb{C}^+ -action on \mathbb{C}^2 induced by \mathbf{g} . Since \mathbb{C}^+ has no non-trivial character, a is an invariant. Hence, $\text{id} \neq a \cdot \mathbf{g} \in \text{Iner}(\mathbb{C}^2, \Gamma)$.

$c) \Rightarrow d)$: Let $\mathbf{g} \in \text{Iner}(\mathbb{C}^2, \Gamma)$ with $\mathbf{g} \neq \text{id}$. By Theorem 3.0.2, \mathbf{g} preserves a trivial \mathbb{C} -bundle $f: \mathbb{C}^2 \rightarrow \mathbb{C}$. Let $\Gamma_i, i \in I$ be the irreducible components of the reduced scheme Γ_{red} . If every Γ_i lies in a fiber of f , then Γ is a fence and thus $\text{Aut}(\mathbb{C}^2, \Gamma)$ is not an algebraic group. Therefore we can assume that $f(\Gamma_i) \subseteq \mathbb{C}$ is dense for some i . As \mathbf{g} is the identity on Γ_i , it follows that \mathbf{g} maps each fiber on itself. Hence, there exists $\alpha \in \mathbb{C}^*$ and a polynomial $b(y)$ such that for each $y \in \mathbb{C}$ the restriction of \mathbf{g} to the fiber $f^{-1}(y)$ is given by

$$\mathbf{g}_y: \mathbb{C} \rightarrow \mathbb{C}, \quad x \mapsto \alpha x + b(y).$$

As \mathbf{g} is the identity on Γ_i , it follows that \mathbf{g}_y has a fixed point for all $y \in f(\Gamma_i)$. If $\alpha = 1$, then \mathbf{g}_y is the identity map for all $y \in f(\Gamma_i) \subseteq \mathbb{C}$. Since $f(\Gamma_i)$ is dense in \mathbb{C} we get a contradiction to the fact that $\mathbf{g} \neq \text{id}$. Thus, $\alpha \neq 1$. But this implies that \mathbf{g}_y has exactly one fixed point for each $y \in \mathbb{C}$. Thus, $\Gamma_i = V(ax + b(y) - x) \simeq \mathbb{C}$ and it is the only irreducible component of Γ_{red} . Therefore, Γ is again a fence and $\text{Aut}(\mathbb{C}^2, \Gamma)$ is not an algebraic group.

$d) \Rightarrow a)$: Assume that Γ is not a fence. By [BS13, Theorem 1], $\text{Aut}(\mathbb{C}^2, \Gamma_{\text{red}})$ is an algebraic group. Now, $\text{Aut}(\mathbb{C}^2, \Gamma)$ is an algebraic group as well, since it is a closed subgroup of $\text{Aut}(\mathbb{C}^2, \Gamma_{\text{red}})$.

□

4. SOME BASIC PROPERTIES OF LOCALLY NILPOTENT DERIVATIONS

Let A be a \mathbb{C} -algebra and assume it is a unique factorization domain (UFD). Let B be a locally nilpotent derivation of A . We call B *irreducible*, if $B \neq 0$ and the following holds: if $B = fB'$ for some locally nilpotent derivation B' and some $f \in \ker B$, then $f \in A^*$ where A^* denotes the subgroup of units of A . Remark, if $A = \mathcal{O}(\mathbb{C}^n)$, then a unipotent automorphism $\mathbf{u} = \text{Exp}(D)$ is irreducible if and only if D is irreducible.

We list some basic facts about locally nilpotent derivations, that we will use constantly (see [Fre06] for proofs).

Lemma 4.0.4. *Let A be a \mathbb{C} -algebra and assume it is a UFD, and let B be a locally nilpotent derivation of A . Then*

- i) *The units of A lie in $\ker B$. In particular, $\mathbb{C} \subseteq \ker B$.*
- ii) *The kernel $\ker B$ is factorially closed in A , i.e. if $f, g \in A$ such that $fg \in \ker B$, then $f, g \in \ker B$.*
- iii) *If $S \subseteq \ker B$ is a multiplicative system, then B extends uniquely to a locally nilpotent derivation of the localization A_S .*
- iv) *If $B \neq 0$, then there exists $f \in A$ such that $B(f) \in \ker B$ and $B(f) \neq 0$.*
- v) *If $s \in A$ such that $B(s) = 1$, then A is a polynomial ring in s over $\ker B$ and $B = \partial/\partial s$.*
- vi) *For $f \in A$, the derivation fB is locally nilpotent if and only if $f \in \ker B$.*
- vii) *If B is irreducible and E is another locally nilpotent derivation of A such that $E(\ker B) = 0$, then there exists $f \in \ker B$ such that $E = fB$.*
- viii) *If $B \neq 0$, then there exists a unique irreducible locally nilpotent derivation B' (up to multiplication by some element of A^*) such that $\ker B = \ker B'$.*
- ix) *If $B(f) \in fA$, then $B(f) = 0$.*
- x) *The exponential $\exp(B) = \sum_{i=0}^{\infty} B^i/i!$ is a \mathbb{C} -algebra automorphism of A and the map \exp defines an injection from the set of locally nilpotent derivations of A to the set of \mathbb{C} -algebra automorphisms of A .*

5. STRUCTURE THEOREMS FOR $\text{Cent}(\mathbf{u})$

5.1. The first unipotent subgroup in $\text{Cent}(\mathbf{u})$. Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. There exists an obvious subgroup of unipotent automorphisms in $\text{Cent}(\mathbf{u})$: The modifications of \mathbf{u}' , i.e. the subgroup $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$. This subgroup has another characterization:

Proposition 5.1.1. *Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent with standard decomposition $\mathbf{u} = d \cdot \mathbf{u}'$. The subgroup $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ consists of those automorphisms of \mathbb{C}^3 that commute with \mathbf{u} and that induce the identity on $\mathbb{C}^3 // \mathbb{C}^+$, i.e. the sequence*

$$1 \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \text{Cent}(\mathbf{u}) \xrightarrow{p} \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma, \Gamma')$$

is exact, where Γ, Γ' denote the plinth divisors of \mathbf{u}, \mathbf{u}' respectively. Moreover, the homomorphisms in the sequence above are homomorphisms of ind-groups.

This result is an immediate consequence of Remark 5.1.1 and Lemma 5.1.2.

Remark 5.1.1. Choose generators v_1, v_2 of the polynomial ring $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}}$ and choose a \mathbb{C} -linear retraction $r: \mathcal{O}(\mathbb{C}^3) \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}}$. The map $p: \text{Cent}(\mathbf{u}) \rightarrow \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma)$ is a morphism of ind-varieties due to the following commutative diagram

$$\begin{array}{ccccc} \text{End}(\mathbb{C}^3) & \xrightarrow[\text{morph.}]{\mathbf{g} \mapsto (\mathbf{g}^*(v_1), \mathbf{g}^*(v_2))} & \mathcal{O}(\mathbb{C}^3)^2 & \xrightarrow[\text{lin.}]{r \times r} & (\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}})^2 \\ \uparrow \text{loc. closed} & & & & \uparrow \text{loc. closed} \\ \text{Cent}(\mathbf{u}) & \xrightarrow{p} & & & \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma, \Gamma'). \end{array}$$

Lemma 5.1.2. *Let A be a \mathbb{C} -algebra and assume it is a UFD, let B, B' be non-zero locally nilpotent derivations of A such that B' is irreducible and $\ker B = \ker B'$. If $\varphi: A \rightarrow A$ is a \mathbb{C} -algebra automorphism, then we have*

$$\varphi|_{\ker B} = \text{id} \text{ and } \varphi \circ B = B \circ \varphi \text{ if and only if } \varphi = \exp(fB'), \quad f \in \ker B.$$

Proof. Assume that $\varphi|_{\ker B}$ is the identity and φ commutes with B . There exists $0 \neq d \in \ker B$, such that $A_d = \ker(B)_d[s]$ is a polynomial ring in an element $s \in A_d$ and $B(s) = 1$, if we extend B to A_d . Since φ commutes with B there exists $g \in \ker(B)_d$ such that the extension $\tilde{\varphi}$ to A_d of φ satisfies $\tilde{\varphi}(s) = s + g$. Now, we have $\varphi = \exp(gB)|_A$. A density argument shows that $\exp(tgB)(A) \subseteq A$ for all $t \in \mathbb{C}$. Since

$$gB = \left. \frac{\exp(tgB) - \text{id}}{t} \right|_{t=0},$$

we have $gB(A) \subseteq A$. Hence gB is a locally nilpotent derivation of A that vanishes on $\ker B = \ker B'$. Thus, $gB = fB'$ for some $f \in \ker B$. The converse is clear. \square

Remark 5.1.2. By Proposition 5.1.1, $\text{Cent}(\mathbf{u})$ normalizes $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ and one can easily see, that the action is given by

$$\mathbf{g}^{-1} \circ f \cdot \mathbf{u}' \circ \mathbf{g} = \mu(\mathbf{g})\mathbf{g}^*(f) \cdot \mathbf{u}', \quad \mathbf{g} \in \text{Cent}(\mathbf{u}), \quad f \in \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'}$$

where $\mu: \text{Cent}(\mathbf{u}) \rightarrow \mathbb{C}^*$ is the homomorphism given by $\mu(\mathbf{g})d = \mathbf{g}^*(d)$.

5.2. Centralizer of a modified translation in $\text{Aut}(\mathbb{C}^3)$.

Proposition 5.2.1. *Let $\text{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be a modified translation with standard decomposition $\mathbf{u} = d \cdot \mathbf{u}'$. Denote by Γ the plinth divisor of \mathbf{u} . Then*

$$1 \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \text{Cent}(\mathbf{u}) \xrightarrow{p} \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma) \rightarrow 1$$

is a split short exact sequence of ind-groups. Moreover there exists a closed subgroup of $\text{Cent}(\mathbf{u})$ that is mapped via p isomorphically onto $\text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma)$.

Remark 5.2.1. Under the assumptions of Proposition 5.2.1, $\text{Cent}(\mathbf{u})$ consists only of algebraic elements, provided that Γ is non-empty. Indeed, let $H \subseteq \text{Cent}(\mathbf{u})$ be a closed subgroup, such that p induces an isomorphism $H \simeq \text{Aut}(\mathbb{C}^3 // \mathbb{C}^+, \Gamma)$. Let $(f \cdot \mathbf{u}', \mathbf{h}) \in \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \rtimes H \simeq \text{Cent}(\mathbf{u})$. By Proposition 3.0.1 i), $R = \langle \mathbf{h} \rangle$ is an algebraic subgroup of H . Hence

$$W = \text{span}\{\mathbf{r}^*(f) \mid \mathbf{r} \in R\}$$

is a finite dimensional subspace of $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'}$ and $W \cdot \mathbf{u}' \rtimes R \subseteq \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \rtimes H$ is an algebraic subgroup that contains $(f \cdot \mathbf{u}', \mathbf{h})$ (see Remark 5.1.2).

Proof of Proposition 5.2.1. By Proposition 5.1.1, the sequence is left exact. By assumption, there exist coordinates (x, y, z) on \mathbb{C}^3 such that $d \in \mathbb{C}[y, z]$ and $\mathbf{u} = (x + d, y, z)$. Moreover, we can identify the algebraic quotient $\pi: \mathbb{C}^3 \rightarrow \mathbb{C}^2$ with the map $(x, y, z) \mapsto (y, z)$ and $\Gamma = \text{div}(d)$. Let $\mathbf{f} \in \text{Aut}(\mathbb{C}^2, \Gamma)$. Then $\mathbf{f}^*(d) = \lambda(\mathbf{f})d$ for some $\lambda(\mathbf{f}) \in \mathbb{C}^*$. One can see that $\text{Aut}(\mathbb{C}^2, \Gamma) \rightarrow \mathbb{C}^*$, $\mathbf{f} \mapsto \lambda(\mathbf{f})$ is a homomorphism of ind-groups. Thus

$$H = \{\sigma \in \text{Aut}(\mathbb{C}^3) \mid \sigma(x, y, z) = (\lambda x, \mathbf{f}(y, z)) \text{ with } \mathbf{f} \in \text{Aut}(\mathbb{C}^2, \Gamma) \text{ and } \lambda(\mathbf{f}) = \lambda\}$$

is a closed subgroup of $\text{Cent}(\mathbf{u})$ (note that the subgroup $\text{Aut}(\mathbb{C}^2, \Gamma) \subseteq \text{Aut}(\mathbb{C}^2)$ is closed by Proposition 3.0.1) and $p|_H: H \rightarrow \text{Aut}(\mathbb{C}^2, \Gamma)$ is an isomorphism of ind-groups. \square

Proposition 5.2.2. *Let $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent. Then $\text{Cent}(\mathbf{u})$ contains $(\mathbb{C}^*)^2$ as an algebraic subgroup if and only if \mathbf{u} is a modified translation and the plinth divisor Γ is given by $v^i w^j$ for some coordinates (v, w) of $\mathbb{C}^3 // \mathbb{C}^+ \simeq \mathbb{C}^2$.*

Proof. Assume that $\text{Cent}(\mathbf{u})$ contains an algebraic subgroup $T \simeq (\mathbb{C}^*)^2$. By Proposition 5.2.1 it follows that T acts faithfully on $\mathbb{C}^3 // \mathbb{C}^+$ and leaves the plinth divisor Γ invariant. Hence, it follows from [BB66] that there exist coordinates (v, w) on $\mathbb{C}^2 \simeq \mathbb{C}^3 // \mathbb{C}^+$ such that Γ is given by $v^i w^j$ for some integers i, j . By [BB67] there exist coordinates (x_1, x_2, x_3) of \mathbb{C}^3 such that the action of T is diagonal with respect to these coordinates. Hence there exist characters $\lambda_1, \lambda_2, \lambda_3$ of T such that $\mathbf{t}(x_1, x_2, x_3) = (\lambda_1(\mathbf{t})x_1, \lambda_2(\mathbf{t})x_2, \lambda_3(\mathbf{t})x_3)$ for all $\mathbf{t} \in T$. Let $\mathbf{u} = \text{Exp}(D)$. By assumption we have for all $\mathbf{t} \in T$ and $i = 1, 2, 3$

$$D(x_i) \circ \mathbf{t} = \lambda_i(\mathbf{t})D(x_i). \quad (3)$$

As the action of T on \mathbb{C}^3 is faithful, the subgroup spanned by $\lambda_1, \lambda_2, \lambda_3$ inside the characters of T has rank 2. Assume first that the λ_i are pairwise different. Then there exist at least two different indices $k_1, k_2 \in \{1, 2, 3\}$ such that λ_{k_i} lies not in the monoid spanned by $\{\lambda_l \mid l \neq k_i\}$. By symmetry we can assume $k_1 = 1, k_2 = 2$. This implies that $D(x_i) \in x_i \mathbb{C}[x_1, x_2, x_3]$ for $i = 1, 2$. Since D is locally nilpotent we have $D(x_1) = D(x_2) = 0$. Hence, \mathbf{u} is a modified translation. Assume now that $\lambda_1 = \lambda_2 \neq \lambda_3$ (the other cases follow by symmetry). Thus, λ_3 does not

lie in the monoid spanned by λ_1 and λ_2 . Hence we get $D(x_3) \in x_3 \mathbb{C}[x_1, x_2, x_3]$ and $D(x_1), D(x_2) \in \mathbb{C}x_1 \oplus \mathbb{C}x_2$. Since D is locally nilpotent it follows that $D(x_3) = 0$ and the linear endomorphism $D|_{\mathbb{C}x_1 \oplus \mathbb{C}x_2}$ is nilpotent. This implies that \mathbf{u} is a modified translation.

The converse of the statement is clear. \square

5.3. The second unipotent subgroup in $\text{Cent}(\mathbf{u})$. Let $\text{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent with standard decomposition $\mathbf{u} = d \cdot \mathbf{u}'$. Throughout this subsection we assume that the plinth divisor $\Gamma = \text{div}(a)$ of \mathbf{u} is a fence. There exists another subgroup of unipotent automorphisms inside $\text{Cent}(\mathbf{u})$ in addition to $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$, that we describe in this subsection.

Lemma 5.3.1. *Let $\text{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent. If the plinth divisor $\Gamma = \text{div}(a)$ is a fence, then there exists a variable z of $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}} = \mathcal{O}(\mathbb{C}^2)$ such that $a \in \mathbb{C}[z]$ and any such z is a variable of $\mathcal{O}(\mathbb{C}^3)$.*

Proof. By Proposition 3.0.3 there exists a coordinate system (z, w) of \mathbb{C}^2 such that the embedding $\text{div}(a) = \Gamma \subseteq \mathbb{C}^2$ is given by the standard embedding $F \times \mathbb{C} \subseteq \mathbb{C}^2$ for some 0-dimensional closed subscheme F of \mathbb{C} . Thus $a \in \mathbb{C}[z]$. Since π is a trivial \mathbb{C} -bundle over $\mathbb{C}^2 \setminus \Gamma$, it follows that only finitely many fibers of $z: \mathbb{C}^3 \rightarrow \mathbb{C}$ are non-isomorphic to \mathbb{C}^2 . Thus z is a variable of $\mathcal{O}(\mathbb{C}^3)$, according to KALIMAN's Theorem [DK09, Theorem 3] (cf. also [BEHEK08, Theorem 3.1]). \square

Remark 5.3.1. If $\mathbf{u} = \text{Exp}(D)$ is irreducible, then Γ is a fence if and only if $\text{rank } D \leq 2$ (i.e. there exists a variable z of $\mathcal{O}(\mathbb{C}^3)$ that lies in $\ker D$). This follows from the lemma above and from [DF98, Theorem 2.4, Proposition 2.3].

Definition 5.3.2. Let A be a UFD and let $P \in A[x, y]$. We denote

$$\Delta_P = -P_y \frac{\partial}{\partial x} + P_x \frac{\partial}{\partial y}$$

where P_x and P_y denote the partial derivatives of P with respect to x and y respectively. Obviously, Δ_P is an A -derivation of $A[x, y]$ and $\Delta_P(P) = 0$.

Let D, D' be locally nilpotent derivations of $\mathcal{O}(\mathbb{C}^3)$ such that $\mathbf{u} = \text{Exp}(D)$ and $\mathbf{u}' = \text{Exp}(D')$. Let $z \in \ker D$ be a variable such that $a \in \mathbb{C}[z]$ and let (x, y, z) be a coordinate system of $\mathcal{O}(\mathbb{C}^3)$ (see Lemma 5.3.1). Let $A = \mathbb{C}[z]$. It follows now from [DF98, Theorem 2.4] that there exists $P \in A[x, y]$ such that

$$D' = \Delta_P \quad \text{and} \quad \ker D = \ker D' = \mathbb{C}[z, P].$$

Obviously, d divides a in $\mathbb{C}[z]$. Let $a = da'$. An easy calculation shows that $\text{div}(a')$ is the plinth divisor of \mathbf{u}' and that for all $Q \in A[x, y]$ we have

$$D(Q) = a \quad \text{if and only if} \quad D'(Q) = a'. \quad (4)$$

By assumption $\Gamma = \text{div}(a)$ is a fence and thus $a, a' \neq 0$.

Lemma 5.3.2. *Let $A = \mathbb{C}[z]$. If $Q \in A[x, y]$ such that $D(Q) = a$, then $E = \Delta_Q$ is an irreducible locally nilpotent derivation. Moreover, E commutes with D .*

Proof. Let K be the quotient field of A . The extension of D' to $K[x, y]$ satisfies $D'(Q/a') = 1$. Thus $K[x, y] = K[P, Q]$. E is non-zero, since $E(P) = -a' \neq 0$. If we extend E to a derivation of $K[x, y]$ one easily sees that E is locally nilpotent. Thus E is a non-zero locally nilpotent derivation of $A[x, y]$.

By [DF98, Theorem 2.4, Proposition 2.3]) there exists $S \in A[x, y]$ and $0 \neq h \in A[P]$ such that $E = h\Delta_S$ and Δ_S is irreducible. Thus $-a' = E(P) = h\Delta_S(P) = -h\Delta_P(S)$. Hence $\Delta_P(S)$ lies in the plinth ideal of Δ_P and thus $\Delta_P(S)$ is a multiple of a' . This implies that $h \in \mathbb{C}^*$ and proves that E is irreducible.

If we extend Δ_P and Δ_Q to $K[x, y] = K[P, Q]$, we get $\Delta_P = a'(\partial/\partial Q)$ and $\Delta_Q = -a'(\partial/\partial P)$. Thus E commutes with D' . Since $d \in \mathbb{C}[z]$, E and $D = dD'$ commute. \square

Definition 5.3.3. For any $Q \in \mathcal{O}(\mathbb{C}^3)$ with $D(Q) = a$ we call

$$\mathbf{e} = \text{Exp}(E) = \text{Exp}(\Delta_Q)$$

an *admissible complement* to \mathbf{u} .

By (4), we get that \mathbf{e} is an admissible complement to \mathbf{u} if and only if \mathbf{e} is an admissible complement to \mathbf{u}' . It follows from Lemma 5.3.2 that $\mathcal{O}(\mathbb{C}^3)^{(\mathbf{e}, \mathbf{u})} \cdot \mathbf{e}$ is a subgroup of unipotent automorphisms inside $\text{Cent}(\mathbf{u})$.

Remark 5.3.4. We have $\mathbb{C}^2 \setminus \Gamma = U \times \mathbb{C}$ for some non-empty open subset $U \subseteq \mathbb{C}$. The restriction of \mathbf{u} and of \mathbf{e} to the open subset

$$\pi^{-1}(\mathbb{C}^2 \setminus \Gamma) = (U \times \mathbb{C}) \times \mathbb{C} = \text{Spec}(\mathbb{C}[z]_a[P, Q])$$

are given by $(u, v, w) \mapsto (u, v, w+1)$ and $(u, v, w) \mapsto (u, v+1, w)$ respectively, where (u, v, w) is the coordinate system $(z, -P/a', Q/a)$.

5.4. The property (Sat). We introduce in this subsection a property for a subset $S \subseteq \text{Aut}(\mathbb{C}^n)$ and we will show that $\text{Cent}(\mathbf{v})$ satisfies this property for any unipotent automorphism $\mathbf{v} \in \text{Aut}(\mathbb{C}^n)$. This property will then play a key role when we describe the set of unipotent elements inside the centralizer. One can think of this property as a saturation feature on the unipotent elements in S .

Definition 5.4.1. Let $S \subseteq \text{Aut}(\mathbb{C}^n)$ be a subset. We say that S has the *property (Sat)* if for all unipotent $\mathbf{w} \in \text{Aut}(\mathbb{C}^n)$ and for all $0 \neq f \in \mathcal{O}(\mathbb{C}^n)^{\mathbf{w}}$ we have

$$f \cdot \mathbf{w} \in S \implies \mathbf{w} \in S. \quad (\text{Sat})$$

Proposition 5.4.1. *If $\mathbf{v} \in \text{Aut}(\mathbb{C}^n)$ is unipotent, then the subgroup $\text{Cent}(\mathbf{v}) \subseteq \text{Aut}(\mathbb{C}^n)$ satisfies the property (Sat).*

Proof. Let $\mathbf{v} = \text{Exp}(B)$ and let $\mathbf{w} = \text{Exp}(F)$. Assume that $f \cdot \mathbf{w}$ commutes with \mathbf{v} for some \mathbf{w} -invariant $0 \neq f \in \mathcal{O}(\mathbb{C}^n)$. If $\mathbf{v} = \mathbf{id}$ or $\mathbf{w} = \mathbf{id}$, then (Sat) is obviously satisfied. Thus we assume $\mathbf{v} \neq \mathbf{id} \neq \mathbf{w}$. For the Lie-bracket we have

$$0 = [fF, B] = f[F, B] - B(f)F. \quad (5)$$

Thus, it is enough to prove that $B(f) = 0$.

First, assume that F is irreducible. By (5), it follows that f divides $B(f)F(g)$ for all $g \in \mathcal{O}(\mathbb{C}^n)$. As F is irreducible, it follows that f divides $B(f)$. Since B is locally nilpotent, it follows that $B(f) = 0$.

Now, let $F = f'F'$ for some irreducible F' . Thus, $ff'F'$ commutes with B and by the argument above, $B(ff') = 0$. Since $\ker B$ is factorially closed in $\mathcal{O}(\mathbb{C}^n)$, we have $B(f) = 0$. \square

5.5. The subgroup $N \subseteq \text{Cent}(\mathbf{u})$. Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent. We define in this subsection a subgroup N of $\text{Cent}(\mathbf{u})$ and we gather some facts about this group. In the next subsection, we will prove that N is exactly the set of unipotent automorphisms $\text{Cent}(\mathbf{u})$ if \mathbf{u} is not a translation.

Definition 5.5.1. Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent with standard decomposition $\mathbf{u} = d \cdot \mathbf{u}'$ and let Γ be the plinth divisor of \mathbf{u} . Let

$$N = N(\mathbf{u}) = \begin{cases} \mathcal{O}(\mathbb{C}^3)^{(\mathbf{e}, \mathbf{u}')} \cdot \mathbf{e} \circ \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' & \text{if } \Gamma \text{ is a fence} \\ \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' & \text{otherwise.} \end{cases}$$

where \mathbf{e} is an admissible complement to \mathbf{u} (cf. subsec. 5.3). Moreover, let

$$M = M(\mathbf{u}) = \begin{cases} (\ker E \cap \ker D')E + \ker(D')D' & \text{if } \Gamma \text{ is a fence,} \\ \ker(D')D' & \text{otherwise.} \end{cases}$$

where $\mathbf{u}' = \text{Exp}(D')$ and $\mathbf{e} = \text{Exp}(E)$ (cf. Lemma 5.3.2).

Proposition 5.5.1. *Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and assume it is not a translation. Then:*

- i) N consists of unipotent automorphisms and we have $N = \text{Exp}(M)$.
- ii) N normalizes $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ and we have for all $\mathbf{g} \in N$ and for all $f \in \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'}$

$$\mathbf{g}^{-1} \circ f \cdot \mathbf{u}' \circ \mathbf{g} = \mathbf{g}^*(f) \cdot \mathbf{u}'.$$

iii) N is a closed normal subgroup of $\text{Cent}(\mathbf{u})$ that fits into the following split short exact sequence of ind-groups

$$1 \rightarrow \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow N \xrightarrow{p|_N} \text{Aut}(\mathbb{C}^2, \Gamma) \cap \text{Iner}(\mathbb{C}^2, \Gamma')_u \rightarrow 1,$$

where Γ' is the plinth divisor of \mathbf{u}' . If Γ is a fence, then the restriction of p to $\mathcal{O}(\mathbb{C}^3)^{(\mathbf{u}', \mathbf{e})} \cdot \mathbf{e}$ is an isomorphism of ind-groups. In particular, N is independent of the choice of \mathbf{e} .

iv) $N \subseteq \text{Aut}(\mathbb{C}^3)$ satisfies the property (Sat).

Proof. Assume first that Γ is not a fence. Then i) ii) and iv) are clear, iii) follows from Proposition 3.0.1. Thus we can assume that Γ is a fence.

- i) Let $hE + fD' \in M$. By induction on $l \geq 1$ one sees that $(hE + fD')^l$ is a sum of terms of the form $gE^i(D')^j$ where $g \in \ker D'$. From this fact, one can deduce that $hE + fD'$ is locally nilpotent and hence M consists only of locally nilpotent derivations.

For all $f \in \ker D'$ and $h \in \ker D' \cap \ker E$ and $q \geq 0$ we have

$$fD' \text{ad}(hE)^q = (-1)^q h^q E^q(f)D'$$

where $A \text{ad}(B) = [A, B]$. With the aid of this formula, an application of the Baker-Campbell-Hausdorff formula yields $\text{Exp}(hE) \circ \text{Exp}(fD') \in M$ (see [Jac62, Proposition 1, §5, chp. V]). Hence $N \subseteq \text{Exp}(M)$ which shows in particular, that N consists of unipotent automorphisms. Moreover, $\exp hE$ and $\exp(fD' + hE)$ coincide on $\ker D'$. Lemma 5.1.2 implies $(\text{Exp } hE)^{-1} \circ \text{Exp}(fD' + hE) = \text{Exp}(gD')$ for some $g \in \ker D'$ and thus $\text{Exp}(M) \subseteq N$.

ii) This follows from Remark 5.1.2.

iii) One can check that $N = p^{-1}(\text{Aut}(\mathbb{C}^2, \Gamma) \cap \text{Iner}(\mathbb{C}^2, \Gamma')_u)$ by using Proposition 5.1.1. Since $\text{Aut}(\mathbb{C}^2, \Gamma) \cap \text{Iner}(\mathbb{C}^2, \Gamma')_u$ is a closed normal subgroup of $\text{Aut}(\mathbb{C}^2, \Gamma, \Gamma')$ it follows that N is a closed normal subgroup of $\text{Cent}(\mathbf{u})$.

It is enough to show that the homomorphism $\mathcal{O}(\mathbb{C}^3)^{(\mathbf{e}, \mathbf{u}')} \cdot \mathbf{e} \rightarrow \text{Aut}(\mathbb{C}^2, \Gamma) \cap \text{Iner}(\mathbb{C}^2, \Gamma')_u$ (induced by p) is an isomorphism of ind-groups. Injectivity follows from the fact that $\mathcal{O}(\mathbb{C}^3)^{(\mathbf{e}, \mathbf{u}')} \cdot \mathbf{e} \cap \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' = \{\mathbf{id}\}$ and surjectivity follows from a straightforward calculation, by using that Γ is non-empty. The inverse map is clearly a morphism.

iv) Let $0 \neq hE + fD' \in M$. It is enough to prove that

$$\gcd(h, f) = 1 \implies hE + fD' \text{ is irreducible} \quad (6)$$

where the greatest common divisor is taken in the polynomial ring $\ker D' = \mathbb{C}[z, P]$ (we use the notation of subsec. 5.3). Indeed, let $gB = hE + fD' \in M$ for some locally nilpotent derivation $B \neq 0$ and some $0 \neq g \in \ker B$ and let $h = \gcd(h, f)h_0$, $f = \gcd(h, f)f_0$. Thus B vanishes on $\ker(h_0E + f_0D')$ and since $h_0E + f_0D'$ is irreducible, there exists $b \in \ker(h_0E + f_0D')$ such that $B = b(h_0E + f_0D')$. This implies $gb = \gcd(h, f) \in \mathbb{C}[z]$ and therefore $b \in \mathbb{C}[z]$. This shows that $B \in M$.

Let us prove (6). Since E and D' are irreducible (see Lemma 5.3.2) we can assume that h and f both are non-zero. A calculation shows

$$hE + fD' = \Delta_F, \quad F = hQ + fP - \int \left(\frac{\partial f}{\partial P} P \right) dP$$

where the integration is taken inside the polynomial ring $\ker D' = \mathbb{C}[z, P]$ and Δ_F is taken with respect to $A[x, y]$ where $A = \mathbb{C}[z]$. Let $f = \sum_{i=0}^n f_i(z)P^i$. Thus we have

$$fP - \int \left(\frac{\partial f}{\partial P} P \right) dP = \sum_{i=0}^n f_i(z) \left(1 - \frac{i}{i+1} \right) P^{i+1}.$$

Denote this last polynomial by $G \in \mathbb{C}[z, P]$.

Now, assume towards a contradiction that $hE + fD'$ is not irreducible. Hence, we have $hE + fD' = bB$ for some locally nilpotent derivation B and some non-constant $b \in \ker B$. By plugging in P and Q in $hE + fD' = bB$ and using the fact that $\gcd(h, f) = 1$ we see that b divides a' (recall that $D'(Q) = a'$ and $E(P) = -a'$). Hence there exists a root z_0 of a' such that the induced derivation of $\Delta_F = hE + fD'$ on $\mathbb{C}[x, y, z]/(z - z_0) \simeq \mathbb{C}[x, y]$ vanishes. Thus, there exists a constant $c \in \mathbb{C}$ such that

$$h(z_0)Q(x, y, z_0) + \sum_{i=0}^n f_i(z_0) \left(1 - \frac{i}{i+1} \right) P^{i+1}(x, y, z_0) = c. \quad (7)$$

The polynomial $P(x, y, z_0) \in \mathbb{C}[x, y]$ is non-constant, since otherwise $\mathbf{u} = \text{Exp}(\Delta_P)$ would have a two-dimensional fixed point set, contradicting the irreducibility (cf. [Dai07, 2.10]). If $h(z_0) = 0$, then we have $f(z_0, P) = 0$ by (7). Hence $\gcd(h, f) \neq 1$, a contradiction. Thus we can assume $h(z_0) \neq 0$. It follows that $Q + h(z_0)^{-1}(G(z, P) - c)$ is divisible by $z - z_0$ inside $\mathcal{O}(\mathbb{C}^3)$. Thus,

$$D' \left(\frac{Q + h(z_0)^{-1}(G(z, P) - c)}{z - z_0} \right) = \frac{a'}{z - z_0}.$$

But this contradicts the fact, that a' is a generator of the plinth ideal of D' . \square

5.6. The group $\text{Cent}(\mathbf{u})$ as a semi-direct product. In this subsection, we prove our first main result: There exists an algebraic subgroup $R \subseteq \text{Cent}(\mathbf{u})$ such that $\text{Cent}(\mathbf{u})$ is the semi-direct product of N with R , if \mathbf{u} is not a translation.

Theorem 5.6.1. *Let $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and assume that \mathbf{u} is not a translation. Then the subgroup $N \subseteq \text{Cent}(\mathbf{u})$ is closed and normal, and there exists an algebraic subgroup $R \subseteq \text{Cent}(\mathbf{u})$ such that $\text{Cent}(\mathbf{u}) \simeq N \rtimes R$ as ind-groups. Moreover, all elements of $\text{Cent}(\mathbf{u})$ are algebraic.*

We prove the result for modified translations and reduce the general case to it.

Proof for a modified translation. Let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. There exists a coordinate system (x, y, z) such that $\mathbf{u}'(x, y, z) = (x + 1, y, z)$ and $d \in \mathbb{C}[y, z] \setminus \mathbb{C}$.

If Γ is not a fence, then it follows from Proposition 3.0.1 that $\text{Aut}(\mathbb{C}^2, \Gamma)$ is an algebraic group. By Proposition 5.2.1 there exists a closed subgroup R of $\text{Cent}(\mathbf{u})$ that is mapped via $p: \text{Cent}(\mathbf{u}) \rightarrow \text{Aut}(\mathbb{C}^2, \Gamma)$ isomorphically onto $\text{Aut}(\mathbb{C}^2, \Gamma)$ and $\text{Cent}(\mathbf{u}) \simeq N \rtimes R$.

Now, assume that $\Gamma = \text{div}(a)$ is a non-empty fence. By Proposition 3.0.3 there exist coordinates (y, z) of $\mathbb{C}^2 = \mathbb{C}^3 // \mathbb{C}^+$ such that $a \in \mathbb{C}[z]$. Thus we have a split short exact sequence of ind-groups

$$1 \rightarrow \text{Aut}(\mathbb{C}^2, \Gamma)_u \hookrightarrow \text{Aut}(\mathbb{C}^2, \Gamma) \xrightarrow{q} \mathbb{C}^* \times \text{Aut}(\mathbb{C}, V(a)) \rightarrow 1$$

where q sends an automorphism $(y, z) \mapsto (\lambda y + h, \alpha z + \beta)$ to $(\lambda, z \mapsto \alpha z + \beta)$. Let R be the algebraic group $\mathbb{C}^* \times \text{Aut}(\mathbb{C}, V(a))$. Since N is generated by $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ and $\mathcal{O}(\mathbb{C}^3)^{(\mathbf{e}, \mathbf{u}')} \cdot \mathbf{e} \simeq \text{Aut}(\mathbb{C}^2, \Gamma)_u$ (see Proposition 5.5.1), we have the desired split short exact sequence of ind-groups

$$1 \rightarrow N \hookrightarrow \text{Cent}(\mathbf{u}) \xrightarrow{q \circ p} R \rightarrow 1.$$

By Remark 5.2.1, every element of $\text{Cent}(\mathbf{u})$ is algebraic. □

Proof in the general case. Let $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}} = \mathbb{C}[\tilde{y}, \tilde{z}]$ and let $\tilde{x} \in \mathcal{O}(\mathbb{C}^3)$ such that $\mathbf{u}^*(\tilde{x}) = \tilde{x} + a$, where $\Gamma = \text{div}(a)$. Let $\tilde{\mathbf{u}} \in \text{Aut}(\mathbb{C}^3)$ be given by $\tilde{\mathbf{u}}(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x} + a, \tilde{y}, \tilde{z})$ where we interpret a as a polynomial in \tilde{y} and \tilde{z} . The morphism $\mathbb{C}^3 \rightarrow \mathbb{C}^3$ induced by the inclusion $\mathbb{C}[\tilde{x}, \tilde{y}, \tilde{z}] \subseteq \mathcal{O}(\mathbb{C}^3)$ is birational and thus we get an injective group homomorphism

$$\eta: \text{Cent}(\mathbf{u}) \longrightarrow \text{Cent}(\tilde{\mathbf{u}}).$$

In fact, η is a homomorphism of ind-groups, due to the following commutative diagram, where $r: \mathcal{O}(\mathbb{C}^3) \rightarrow \mathbb{C}[\tilde{x}, \tilde{y}, \tilde{z}]$ is a \mathbb{C} -linear retraction

$$\begin{array}{ccccc} \mathcal{O}(\mathbb{C}^3)^3 & \xrightarrow[\text{morph.}]{\mathbf{g} \mapsto (\mathbf{g}^*(\tilde{x}), \mathbf{g}^*(\tilde{y}), \mathbf{g}^*(\tilde{z}))} & \mathcal{O}(\mathbb{C}^3)^3 & \xrightarrow[\text{lin.}]{r \times r \times r} & \mathbb{C}[\tilde{x}, \tilde{y}, \tilde{z}]^3 \\ \uparrow \text{loc. closed} & & & & \uparrow \text{loc. closed} \\ \text{Cent}(\mathbf{u}) & \xrightarrow{\eta} & & & \text{Cent}(\tilde{\mathbf{u}}) \end{array}$$

According to the first case, $\text{Cent}(\tilde{\mathbf{u}})$ is the semi-direct product of $N(\tilde{\mathbf{u}})$ with some algebraic subgroup $\tilde{R} \subseteq \text{Cent}(\tilde{\mathbf{u}})$. Let $H \subseteq \tilde{R}$ be an algebraic subgroup. We claim that $\eta^{-1}(H) \subseteq \text{Cent}(\mathbf{u})$ is an algebraic subgroup. Since $\eta: \text{Cent}(\mathbf{u}) \rightarrow \text{Cent}(\tilde{\mathbf{u}})$ is a homomorphism of ind-groups, it follows that $\eta^{-1}(H)$ is a closed subgroup. As H is

algebraic and thus acts locally finite on \mathbb{C}^3 , it follows that $\eta^{-1}(H)$ acts also locally finite on \mathbb{C}^3 by [KS13, Lemma 3.6]. This implies the claim.

According to the claim all elements of $\text{Cent}(\mathbf{u})$ are algebraic and $R = \eta^{-1}(\tilde{R})$ is algebraic as well. Since η is an injective homomorphism of ind-groups we have the following commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & N(\tilde{\mathbf{u}})^{\mathbb{C}} & \longrightarrow & \text{Cent}(\tilde{\mathbf{u}}) & \twoheadrightarrow & \tilde{R} \longrightarrow 1 \\
 \text{iso. of groups} \uparrow & & \uparrow & & \uparrow \eta & & \uparrow \text{cl. embedd.} \\
 1 & \longrightarrow & N(\mathbf{u})^{\mathbb{C}} & \longrightarrow & \text{Cent}(\mathbf{u}) & \twoheadrightarrow & R \longrightarrow 1
 \end{array}$$

As the first column is a split short exact sequence of ind-groups, the second column is also a split short exact sequence of ind-groups. This proves the theorem. \square

5.7. The unipotent elements of $\text{Cent}(\mathbf{u})$. The goal of this subsection is to prove our second main result: The unipotent elements of $\text{Cent}(\mathbf{u})$ are exactly N provided \mathbf{u} is not a translation. As we know from Proposition 5.5.1 the set N satisfies the property (Sat). This will be a key ingredient in the proof.

Theorem 5.7.1. *Let $\mathbf{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and assume it is not a translation. Then the set of unipotent elements of $\text{Cent}(\mathbf{u})$ is equal to N .*

Proof. Let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. Let $\mathbf{g} \in \text{Cent}(\mathbf{u})$ be a unipotent automorphism with $\mathbf{g} \neq \mathbf{id}$. If Γ is not a fence, then $\text{Aut}(\mathbb{C}^2, \Gamma)$ contains no unipotent automorphism $\neq \mathbf{id}$ (see Proposition 3.0.1). By Proposition 5.1.1 it follows that $\mathbf{g} \in \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' = N$.

Hence we can assume that $\Gamma = \text{div}(a)$ is a fence. Let $z \in \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}}$ be a variable of $\mathcal{O}(\mathbb{C}^3)$ such that $a \in \mathbb{C}[z]$ (see Lemma 5.3.1). If $\mathcal{O}(\mathbb{C}^3)^{\mathbf{g}} = \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}}$, then \mathbf{g} is a modification of \mathbf{u}' and therefore $\mathbf{g} \in N$. Now, assume $\mathcal{O}(\mathbb{C}^3)^{\mathbf{g}} \neq \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}}$. Thus \mathbf{g} is not a modification of \mathbf{u}' and hence $\mathbf{id} \neq p(\mathbf{g}) \in \text{Aut}(\mathbb{C}^2, \Gamma)$. Since $p(\mathbf{g})$ is unipotent and $0 \neq a \in \mathbb{C}[z]$, it follows that $z \in \mathcal{O}(\mathbb{C}^3)^{\mathbf{g}}$ and thus $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle}$ is an ∞ -dimensional \mathbb{C} -vector space. By Theorem 5.6.1 there exists an algebraic subgroup $R \subseteq \text{Cent}(\mathbf{u})$ and a split short exact sequence of ind-groups

$$1 \rightarrow N \hookrightarrow \text{Cent}(\mathbf{u}) \xrightarrow{r} R \rightarrow 1.$$

If $\mathbf{g} \notin N$, then $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle} \cdot \mathbf{g} \cap N = 1$, since N satisfies the property (Sat). We get an injection $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle} \rightarrow \text{Cent}(\mathbf{u}) \twoheadrightarrow R$, $h \mapsto r(h \cdot \mathbf{g})$. Choose any filtration by finite dimensional \mathbb{C} -subspaces to turn $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle}$ into an ind-group. It follows that $\mathcal{O}(\mathbb{C}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle} \rightarrow R$ is an injective homomorphism of ind-groups. But this implies that R has algebraic subgroups of arbitrary high dimension, which is absurd. This finishes the proof of the theorem. \square

If we endow $\text{Cent}(\mathbf{u})/N$ with the algebraic group structure induced by the semi-direct product decomposition coming from Theorem 5.6.1, then we get immediately the following corollary from Theorem 5.7.1.

Corollary 5.7.2. *If $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ is unipotent and not a translation, then the algebraic group $\text{Cent}(\mathbf{u})/N$ consists only of semi-simple elements. In particular, the connected component of the neutral element in $\text{Cent}(\mathbf{u})/N$ is a torus.*

6. APPLICATIONS

Proposition 6.0.3. *Let $\mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent and irreducible, and assume that Γ is a non-empty fence. Then $\Gamma \subseteq \mathbb{C}^3 // \mathbb{C}^+$ is the largest closed subscheme fixed by $\text{Cent}(\mathbf{u})_u$.*

Proof. By Proposition 5.5.1 and Theorem 5.7.1, we get $p(\text{Cent}(\mathbf{u})_u) = \text{Iner}(\mathbb{C}^2, \Gamma)_u$, where $p: \text{Cent}(\mathbf{u}) \rightarrow \text{Aut}(\mathbb{C}^2, \Gamma)$ is the canonical morphism. Thus Γ is fixed by the action of $\text{Cent}(\mathbf{u})_u$. Let $X \subseteq \mathbb{C}^2$ be a closed subscheme that is fixed under $\text{Cent}(\mathbf{u})_u$ and assume that X contains Γ . Moreover, let $I(X) \subseteq \mathcal{O}(\mathbb{C}^2)$ be the vanishing ideal of X and let $\Gamma = \text{div}(a)$. By Proposition 3.0.3 there exist coordinates (z, w) of \mathbb{C}^2 such that $a \in \mathbb{C}[z]$. Let $\sigma \in \text{Iner}(\mathbb{C}^2, \Gamma)_u$ be given by $\sigma(z, w) = (z, w + a)$. By assumption, we get $a = \sigma^*(w) - w \in I(X)$. But this implies that X is a closed subscheme of Γ and hence $X = \Gamma$. \square

Proposition 6.0.4. *Let $\text{id} \neq \mathbf{u} \in \text{Aut}(\mathbb{C}^3)$ be unipotent, not a translation, and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. Then the subgroup $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ of $\text{Cent}(\mathbf{u})$ is characteristic, i.e. $\mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ is invariant under all abstract group automorphisms of $\text{Cent}(\mathbf{u})$.*

Lemma 6.0.5. *Let T be a torus acting on \mathbb{C}^2 . Assume that there exist coordinates (z, w) of \mathbb{C}^2 such that z is a semi-invariant. Then there exists $r \in \mathbb{C}[z]$ such that the action of T is diagonal with respect to the coordinate system $(z, w + r)$.*

Proof of Lemma 6.0.5. By assumption, $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}$, $(z, w) \rightarrow z$ is T -equivariant with respect to a suitable T -action on \mathbb{C} . Due to [KK96, Proposition 1], every lift of a T -action on \mathbb{C} to \mathbb{C}^2 (with respect to π) is equivalent to a trivial lift (with respect to π). Thus, there exists $r \in \mathbb{C}[z]$ such that $w + r$ is a semi-invariant with respect to the action of T . This finishes the proof. \square

Proof of Proposition 6.0.4. Let $R \subseteq \text{Cent}(\mathbf{u})$ be an algebraic subgroup such that $\text{Cent}(\mathbf{u}) = R \ltimes N$ (see Theorem 5.6.1). Let $T = R^0$ be the connected component of the neutral element in R . By Corollary 5.7.2, it is a torus (possibly $\dim T = 0$). It follows from Theorem 5.6.1 that $T \ltimes N \subseteq \text{Cent}(\mathbf{u})$ is a subgroup of finite index. Moreover, $T \ltimes N$ has no proper subgroup of finite index, as this group is generated by groups that have no proper subgroup of finite index. This implies that $T \ltimes N$ is a characteristic subgroup of $\text{Cent}(\mathbf{u})$. Let $G = T \ltimes N$ and let $H = \mathcal{O}(\mathbb{C}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \subseteq N$. It is now enough to prove, that H is characteristic in G . We divide the proof now in two cases.

Γ is not a fence: If $\dim T = 0$, then we have $H = G$. So assume $\dim T > 0$. There exist coordinates (v_1, v_2) of $\mathbb{C}^3 // \mathbb{C}^+$ such that the action of T on $\mathbb{C}^3 // \mathbb{C}^+$ is diagonal with respect to (v_1, v_2) (see [Kam79]). Let ρ_1 and ρ_2 be the characters of T such that $\mathbf{t}^*(v_i) = \rho_i(\mathbf{t})v_i$ for all $\mathbf{t} \in T$. Let $\mathbf{t} \circ f \cdot \mathbf{u}' \in \text{Cent}_G(G^{(1)})$, where $G^{(1)} = [G, G]$ denotes the first derived group. A calculation shows for $i = 1, 2$ and for all $k \geq 0$

$$\text{id} = [\mathbf{t} \circ f \cdot \mathbf{u}', [\mathbf{t}^{-1}, v_i^k \cdot \mathbf{u}']] = (1 - \mu(\mathbf{t}^{-1})\rho_i(\mathbf{t}^{-1})^k)(1 - \mu(\mathbf{t})\rho_i(\mathbf{t})^k)v_i^k \cdot \mathbf{u}'$$

where $\mu: T \rightarrow \mathbb{C}^*$ is the character defined by $\mu(\mathbf{t})d = \mathbf{t}^*(d)$ (see Remark 5.1.2). Thus $\rho_i(\mathbf{t}) = 1$ for $i = 1, 2$. As the action of T on $\mathbb{C}^3 // \mathbb{C}^+$ is faithful, it follows that $\mathbf{t} = 1$. Hence $H = \text{Cent}_G(G^{(1)})$.

Γ is a fence: Let (z, P) be a coordinate system of $\mathbb{C}^2 = \mathbb{C}^3 // \mathbb{C}^+$ such that $\Gamma \subseteq \mathbb{C}^2$ is given by the standard embedding $F \times \mathbb{C} \subseteq \mathbb{C}^2$ for some 0-dimensional closed subscheme $F \subseteq \mathbb{C}$ and $\mathbf{u}' = \text{Exp}(\Delta_P)$ (see subsec. 5.3). As the torus T leaves $\Gamma = V(a) \subseteq \mathbb{C}^2$ invariant and since $a \in \mathbb{C}[z]$, there exists $q \in \mathbb{C}$, such that $z + q$ is a semi-invariant for the action of T . By replacing $z + q$ with z , we can assume that z is a semi-invariant. Moreover, by replacing P with a suitable $P + r$ for some $r \in \mathbb{C}[z]$ we can assume that the action of T with respect to (z, P) is diagonal (see Lemma 6.0.5). Moreover, we denote by \mathbf{e} an admissible complement to \mathbf{u} .

Assume first $\dim T = 0$. Let $h \cdot \mathbf{e} \circ f \cdot \mathbf{u}' \in \text{Cent}_G(G^{(1)})$. A calculation shows that

$$\mathbf{id} = [h \cdot \mathbf{e} \circ f \cdot \mathbf{u}', [P^2 \cdot \mathbf{u}', \mathbf{e}]] = -2h(a')^2 \cdot \mathbf{u}'$$

where $a' \in \mathbb{C}[z]$ such that $\Gamma' = \text{div}(a')$. Hence $h = 0$ and therefore $H = \text{Cent}_G(G^{(1)})$.

Assume now, $\dim T > 0$. Let $A = \mathbb{C}[z]$ and let $A \ltimes A[P]$ be the semi-direct product defined by

$$(h, f) \cdot (\bar{h}, \bar{f}) = (h + \bar{h}, f(P - \bar{h}a') + \bar{f}) \quad (8)$$

From Proposition 5.5.1 it follows that

$$A \ltimes A[P] \xrightarrow{\sim} N(\mathbf{u}), \quad (h, f) \mapsto h \cdot \mathbf{e} \circ f \cdot \mathbf{u}'$$

is an isomorphism of groups. Under this isomorphism the subgroup $A[P]$ is sent onto H . It follows from Lemma 6.0.6 that $H = \text{Cent}_G G^{(2)}$. \square

Lemma 6.0.6. *Let $A \ltimes A[P]$ be defined as in (8) and let $G = T \ltimes (A \ltimes A[P])$ where T is a torus with $\dim T > 0$. Assume that $A[P] \subseteq G$ is a normal subgroup, that the action of T by conjugation on $A[P]$ is given by $\lambda^{-1} f \lambda = \mu(\lambda) f(\rho_1(\lambda)z, \rho_2(\lambda)P)$ for some characters μ, ρ_1, ρ_2 and that $\ker \rho_1 \cap \ker \rho_2 = \{1\}$. Furthermore we assume that the action of T by conjugation on the quotient $G/A[P]$ is non-trivial and the product in G satisfies $(\lambda, 0, 0) \cdot (1, h, f) = (\lambda, h, f)$. Then $A[P] = \text{Cent}_G G^{(2)}$.*

Proof of Lemma 6.0.6. As the action by conjugation of T on $G/A[P]$ is non-trivial, it follows that the first derived subgroup $G^{(1)}$ is not contained in $A[P]$. As T is abelian it follows that $G^{(1)} \subseteq A \ltimes A[P]$ and as A is abelian we conclude $G^{(2)} \subseteq A[P]$. Thus there exists $(1, h_0, f_0) \in G^{(1)}$ with $h_0 \neq 0$. As $A[P]$ is abelian, we get $A[P] \subseteq \text{Cent}_G G^{(2)}$. Now, we prove $\text{Cent}_G G^{(2)} \subseteq A[P]$. We have

$$(1, 0, q - q(P + h_0 a')) = [(1, 0, q), (1, h_0, f_0)] \in G^{(2)} \quad \text{for all } (1, 0, q) \in G^{(1)}.$$

Moreover, $(1, 0, z^i P^j) \in G^{(1)}$ for all $(i, j) \in \mathbb{N}_0^2$ such that $\mu \rho_1^i \rho_2^j$ is not the trivial character, as we have $(1, 0, z^i P^j) = [(\lambda, 0, 0), (1, 0, z^i P^j)]$ for some well chosen $\lambda \in T$. For all $j \geq 0$, the character $\mu \rho_1^i \rho_2^j$ is non-trivial, provided that i is large enough. For all $(1, 0, f) \in A[P]$ we have

$$\begin{aligned} \text{Cent}_G(1, 0, f) &= \{ (\bar{\lambda}, \bar{h}, \bar{f}) \in G \mid (\bar{\lambda}, \bar{h}, \bar{f})^{-1} \cdot (1, 0, f) \cdot (\bar{\lambda}, \bar{h}, \bar{f}) = (1, 0, f) \} \\ &= \{ (\bar{\lambda}, \bar{h}, \bar{f}) \in G \mid f = (\bar{\lambda}^{-1} f \bar{\lambda})(P - \bar{h}a') \} \end{aligned}$$

Let $(\bar{\lambda}, \bar{h}, \bar{f}) \in \text{Cent}_G G^{(2)}$. Since $(\bar{\lambda}, \bar{h}, \bar{f}) \in \text{Cent}_G(1, 0, -z^i h_0 a')$ for i sufficiently large, we get $\bar{\lambda} \in \ker \rho_1$. Moreover, $(\bar{\lambda}, \bar{h}, \bar{f}) \in \text{Cent}_G(1, 0, -z^i h_0 a'(2P + h_0 a'))$ for sufficiently large i . This implies $\rho_2(\bar{\lambda}) = \mu(\bar{\lambda})^{-1}$ and $\bar{h} = ((\mu(\bar{\lambda}) - 1)/2)h_0$. Since $(\bar{\lambda}, \bar{h}, \bar{f}) \in \text{Cent}_G(1, 0, -z^i h_0 a'(3P^2 + 3Ph_0 a' + (h_0 a')^2))$ it follows that $\rho_2(\bar{\lambda})^2 = \mu(\bar{\lambda})^{-1}$. Therefore $\bar{\lambda} \in \ker \rho_2$, $\bar{h} = 0$. Hence we have $(\bar{\lambda}, \bar{h}, \bar{f}) = (1, 0, \bar{f}) \in A[P]$ and this proves $\text{Cent}_G G^{(2)} \subseteq A[P]$. \square

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